

General expressions for extra-dimensional tree amplitudes and all-plus 1-loop integrands in \mathcal{Q} -cut representation

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ABSTRACT: In this paper, we give the general expressions for a special series of tree amplitudes of the Yang-Mills theory. This series of amplitudes have two adjacent massless spin-1 particle with extra-dimensional momenta and any number of positive helicity gluons. With special helicity choices, we use the spinor helicity formalism to express these n-point amplitudes in compact forms, and find a clever way to use the BCFW recursion relations to prove the results. Then these amplitudes are used to compute the 1-loop all-plus integrand with any number of gluons using the \mathcal{Q} -cut construction.

KEYWORDS: BCFW, \mathcal{Q} -cut, Extra Dimension, amplitudes.

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1. Introduction

Recently the \mathcal{Q} -cut construction [1] [2] has been developed to compute complete loop integrands of massless field theory. When using \mathcal{Q} -cut representation to calculate loop integrands, a special series of on-shell tree amplitudes in general D -dimension are required. These tree amplitudes have two adjacent legs with extra-dimensional momenta and other legs with 4-dimensional momenta.

For all-plus 1-loop gluon amplitudes in Yang-Mills theory, the helicity of 4-dimensional particles (gluons) of the required tree amplitudes are all plus, which is just similar to the MHV amplitudes. Our motivation for this paper is to see if we can generalize these tree amplitudes to include any number of positive helicity gluons and find compact expressions for them. With these tree amplitudes we can calculate general expressions for the integrands for n-point 1-loop all-plus amplitudes in \mathcal{Q} -cut representation.

However, it is difficult to do this work by directly calculating a tremendous number of Feynman diagrams involving n gluons. Fortunately, calculations of multi-particle amplitudes have greatly developed

in the last ten years, starting from the twistor string description of $\mathcal{N} = 4$ Yang-Mills proposed by Witten [3]. New ideas have led to the development of new powerful formalisms. The MHV vertex expansion (CSW)[4] and the BCFW recursion relations [5, 6] are two most important ones. Recently CHY formula [7][8] is developed, which can be used to calculate scattering amplitudes in arbitrary dimension.

In this paper, to deal with our extra-dimensional cases, we focus on the BCFW recursion relations, which is an on-shell recursive method developed by Britto, Cachazo, Feng, and Witten, where an higher-point tree amplitude can be given by the sum of products of a propagator and two on-shell lower-point amplitudes with shifted, complex momenta. Later, though the work of Badger, Glover, Khoze, and Svrček [9], the BCFW recursion relations have been generalized to include massive particles with spin.

The BCFW recursion relations for massive particles are also applicable in our cases. But the massless non-scalar particles with D -dimensional momenta are different from massive ones with 4-dimensional momenta: they have more polarization freedom. So this series of extra-dimensional amplitudes contain more kinds of amplitudes than massive cases and we need to choose more helicity for extra-dimensional momenta. In this paper, we follow the similar treatments to use the BCFW recursion relations in amplitudes with massive legs and apply the generalized spinor helicity formalism for massive amplitudes to express amplitudes with extra-dimensional momenta. Our results for this series of different amplitudes are concise and have a common structure astonishingly.

In our cases, spin-1 particles with extra-dimension momenta can be thought as two massive particles with equal mass and 4-dimensional momenta. But whether we should treat any one of two as massive spin-1 particle or massive scalar particle depends on its polarization state. If the polarization state of the extra-dimensional spin-1 particle is 4-dimensional, the particle can be treated as a massive spin-1 particles. If the state is extra-dimensional, the particle can be treated as a massive scalar particle. We follow the generalization of spinor helicity formalism for massive particles to express our results in helicity amplitudes. By introducing a null reference vector q_i , a massive momentum p_i can be decomposed along two lightlike directions. In this paper we set all $q_i = q$, so $p_i = p_i^\perp - \frac{m_i^2}{2q \cdot p_i} q$. The amplitudes are then q -dependent. For an introduction to the approaches to generalize massless spinor helicity formalism, see [10]. In [11, 12, 13], applications of BCFW recursion relations with massive particles can be found. Compact expressions for several towers of tree-level amplitudes with a complex scalar-antiscalar pair or a massive W-boson pair and any numbers of positive helicity gluons are given in [14, 15, 16]. Besides, in [15], superamplitudes on the Coulomb-branch of $N = 4$ SYM are calculates from massive amplitudes. So the massive version of our extra-dimensional amplitudes can be used to study superamplitudes on the Coulomb-branch.

This paper is organized as following. In section 2, we set up conventions and give compact expressions for these tree amplitudes. In section 3, a proof by mathematical induction of the results using the BCFW recursion relations is given. Then in section 4, we use the expressions to give general expressions for the integrands of all-plus 1-loop amplitudes in \mathcal{Q} -cut representation.

2. Conventions and main results

In this section, we will first introduce our conventions and helicity choices. Then, to set our starting point, we give expressions for related 3-point tree amplitudes and 4-point tree amplitudes. Finally we summarize our main results for n -point tree amplitudes and analyze their structures.

We only need to calculate some of amplitudes, since color-ordered amplitudes have the cyclic property $\mathcal{A}_n(12\dots n) = \mathcal{A}_n(2\dots n1)$, and the reflection property: $\mathcal{A}_n(12\dots n) = (-1)^n \mathcal{A}_n(n\dots 21)$.

2.1 Helicity choices and polarization vectors

The convention of D -dimensional metric is $g^{\mu\nu} = \text{diag}(+, -, -, \dots, -)$ throughout the paper. Let's denote an extra-dimensional momentum without label i as

$$\widehat{\ell} = (\ell, \vec{\mu}) \quad , \quad \vec{\mu} = (\mu_1, \dots, \mu_d) \quad , \quad (2.1)$$

where ℓ is the 4-dimensional component and $\vec{\mu}$ is a vector in the extra d -dimension ($d = D - 4$). For the Euclidean d -dimensional space, the extra-dimensional basis e_a for $a = 1, \dots, d$ can be chosen as $\vec{\mu}_{e_a} = (\delta_{1a}, \dots, \delta_{ia}, \dots, \delta_{da})$. The massless condition of $\widehat{\ell}$ is then $\ell^2 - \vec{\mu}^2 = 0$.

By introducing a null auxiliary momentum q in 4-dimension and $q \cdot \ell \neq 0$, we can define the null momentum as

$$\ell^\perp = \ell - \frac{\vec{\mu}^2}{\langle q | \ell | q \rangle} q \quad , \quad (\ell^\perp)^2 = 0 \quad , \quad (2.2)$$

The transverse condition $\epsilon \cdot \widehat{\ell} = 0$ tells that the transverse space is $(D - 2)$ -dimensional. By imposing the extra condition $\epsilon_i \cdot q = 0$, the transverse polarization vectors are fixed. The polarization vectors in [2] are listed here with $D - 2$ helicity choices $+$ $-$ and S_a for spin-1 particles

$$\begin{aligned} \epsilon_\nu^+(\widehat{\ell}, q) &= \left(\frac{\langle q | \gamma_\nu | \ell^\perp \rangle}{\sqrt{2} \langle q | \ell^\perp \rangle}, \vec{0}_d \right) \quad , \quad \epsilon_\nu^-(\widehat{\ell}, q) = \left(\frac{\langle \ell^\perp | \gamma_\nu | q \rangle}{\sqrt{2} [\ell^\perp | q]}, \vec{0}_d \right) \quad , \\ \epsilon^{S_a}(\widehat{\ell}, q) &= \left(\frac{2\mu_a}{\langle q | \ell | q \rangle} q, e_a \right) \quad , \quad a = 1, \dots, d \quad , \end{aligned} \quad (2.3)$$

where e_a is the basis vector of the extra d -dimension, $\vec{0}_d$ denotes the d -dimensional vanishing vector, and μ_a is the a -th component of $\vec{\mu}$ in (2.1). The rest two polarization vectors are longitudinal and time-like which have no physical effect, defined as

$$\epsilon^L(\widehat{\ell}, q) = \widehat{\ell} \quad , \quad \epsilon^T(\widehat{\ell}, q) = (q, \vec{0}_d) \quad . \quad (2.4)$$

These polarization vectors possess the following properties

$$\begin{aligned} \epsilon^\pm \cdot q &= \epsilon^\pm \cdot \ell = \epsilon^\pm \cdot \epsilon^\pm = 0 \quad , \quad \epsilon^+ \cdot \epsilon^- = -1 \quad , \\ \epsilon^L \cdot \epsilon^T &= \langle q | \ell | q \rangle \quad , \quad \epsilon^{S_a} \cdot \epsilon^{L,T} = 0 \quad , \quad \epsilon^{S_a} \epsilon^{S_b} = -\delta^{ab} \quad . \end{aligned} \quad (2.5)$$

The metric can be decomposed as

$$\eta_{\mu\nu} = \frac{\epsilon^L \epsilon^T + \epsilon^T \epsilon^L}{\langle q | \ell | q \rangle} - \epsilon^+ \epsilon^- - \epsilon^- \epsilon^+ - \sum_{a=1}^d \epsilon^{S_a} \epsilon^{S_a} . \quad (2.6)$$

The convention above can be generalized to $(4 - 2\epsilon)$ -dimension trivially, where we should replace $d \rightarrow -2\epsilon$.

In our cases, we have two spin-1 extra-dimensional particles denoted by their 4-dimensional momenta ℓ_2 and ℓ_1 . Since there are D-2 helicity states $+$ $-$ and S_a for each spin-1 particles with extra-dimensional momenta and all gluons have positive helicity, there are 9 kinds of amplitudes distinguished by the helicity of the two extra-dimensional particles. We denote these tree amplitudes as $\{\ell_2^+, \ell_1^+\}$, $\{\ell_2^+, \ell_1^{S_a}\}$, $\{\ell_2^+, \ell_1^-\}$, $\{\ell_2^-, \ell_1^+\}$, $\{\ell_2^-, \ell_1^{S_a}\}$, $\{\ell_2^-, \ell_1^-\}$, $\{\ell_2^{S_a}, \ell_1^+\}$, $\{\ell_2^{S_a}, \ell_1^-\}$, $\{\ell_2^{S_a}, \ell_1^{S_b}\}$.

2.2 3-point amplitudes

Since the 3-point amplitudes perform as building blocks in the BCFW recursion relations, we should calculate 3-point amplitudes first. Here we use simple Feynman rules to obtain

$$\mathcal{A}(1, 2, 3) = -\sqrt{2} [(p_2 \cdot \epsilon_3)(\epsilon_1 \cdot \epsilon_2) + (p_3 \cdot \epsilon_1)(\epsilon_2 \cdot \epsilon_3) + (p_1 \cdot \epsilon_2)(\epsilon_3 \cdot \epsilon_1)] . \quad (2.7)$$

To use the BCFW recursion relations in our cases, we only need these 3-point amplitudes: $\mathcal{A}_3(1^+, \ell_2^+, \ell_1^+)$, $\mathcal{A}_3(1^+, \ell_2^+, \ell_1^-)$, $\mathcal{A}_3(1^+, \ell_2^+, \ell_1^{S_a})$, $\mathcal{A}_3(1^+, \ell_2^-, \ell_1^-)$, $\mathcal{A}_3(1^+, \ell_2^-, \ell_1^{S_a})$, $\mathcal{A}_3(1^+, \ell_2^{S_a}, \ell_1^{S_b})$. Here we choose 1 to represent the particle in 4-dimension and ℓ_1 , ℓ_2 to represent the particles in D-dimension. Then we use the transverse polarization vectors given in (2.3) to give specific expressions. When we use (2.7), we must pay attention to $(p \cdot \epsilon)$. If the momentum p has extra-dimensional components and $\epsilon = \epsilon^{S_a}$ ($a = 1, 2, \dots, d$), the product gets contribution from d-dimensional momentum component: $\epsilon^{S_a}(\widehat{\ell}_1, q) \cdot p_{\widehat{\ell}_2} = \frac{\mu_{\ell_1, a} \langle q | \ell_2^\perp | q \rangle}{\langle q | \ell_1^\perp | q \rangle} - \mu_{\ell_2, a}$.

The expressions for 3-point amplitudes are

$$\begin{aligned} \mathcal{A}_3(1^+, \ell_2^+, \ell_1^+) &= \mathcal{A}_3(1^+, \ell_2^+, \ell_1^{S_a}) = 0 , \\ \mathcal{A}_3(1^+, \ell_2^+, \ell_1^-) &= \frac{[1 \ell_2^\perp]^3}{[\ell_2^\perp \ell_1^\perp][\ell_1^\perp 1]} , \quad \mathcal{A}_3(1^+, \ell_2^-, \ell_1^-) = \frac{\langle \ell_2^\perp \ell_1^\perp \rangle^3}{\langle \ell_1^\perp 1 \rangle \langle 1 \ell_2^\perp \rangle} , \\ \mathcal{A}_3(1^+, \ell_2^-, \ell_1^{S_a}) &= \frac{\sqrt{2} \mu_{\ell_1, a} \langle \ell_2^\perp q \rangle \langle \ell_1^\perp \ell_2^\perp \rangle^2}{\langle \ell_1^\perp q \rangle \langle 1 \ell_1^\perp \rangle \langle 1 \ell_2^\perp \rangle} , \quad \mathcal{A}_3(1^+, \ell_2^{S_a}, \ell_1^-) = -\frac{\sqrt{2} \mu_{\ell_2, a} \langle \ell_1^\perp q \rangle \langle \ell_1^\perp \ell_2^\perp \rangle^2}{\langle \ell_2^\perp q \rangle \langle 1 \ell_1^\perp \rangle \langle 1 \ell_2^\perp \rangle} \\ \mathcal{A}_3(1^+, \ell_2^{S_a}, \ell_1^{S_b}) &= -\delta^{ab} \frac{[1 \ell_2^\perp][1 \ell_1^\perp]}{[\ell_1^\perp \ell_2^\perp]} , \end{aligned} \quad (2.8)$$

Notice that the reflection property of $\mathcal{A}_3(1^+, \ell_2^{S_a}, \ell_1^-) = (-1)^3 \mathcal{A}_3(1^+, \ell_1^-, \ell_2^{S_a})$, which is not obvious, is also obeyed because $\mu_{\ell_1, a} = -\mu_{\ell_2, a}$.

2.3 4-point amplitudes

In appendix of [2], the expressions below are calculated using Feynman rules

$$\begin{aligned}
\mathcal{A}_4(1^+, 2^+, \ell_2^+, \ell_1^+) &= \mathcal{A}_4(1^+, 2^+, \ell_2^+, \ell_1^{S_a}) = 0, \\
\mathcal{A}_4(1^+, 2^+, \ell_2^+, \ell_1^-) &= \frac{\mu^2 \langle \ell_1^\perp q \rangle^2}{\langle \ell_2^\perp q \rangle^2} \frac{[2\ 1]}{\langle 1\ 2 \rangle \langle 1 | \ell_1 | 1 \rangle}, \\
\mathcal{A}_4(1^+, 2^+, \ell_2^-, \ell_1^-) &= -\langle \ell_1^\perp \ell_2^\perp \rangle^2 \frac{[2\ 1]}{\langle 1\ 2 \rangle \langle 1 | \ell_1 | 1 \rangle}, \\
\mathcal{A}_4(1^+, 2^+, \ell_2^-, \ell_1^{S_a}) &= \frac{\sqrt{2} \mu_{\ell_1, a} \langle \ell_1^\perp \ell_2^\perp \rangle \langle \ell_2^\perp q \rangle}{\langle \ell_1^\perp q \rangle} \frac{[2\ 1]}{\langle 1\ 2 \rangle \langle 1 | \ell_1 | 1 \rangle}, \\
\mathcal{A}_4(1^+, 2^+, \ell_2^{S_a}, \ell_1^-) &= -\frac{\sqrt{2} \mu_{\ell_2, a} \langle \ell_2^\perp \ell_1^\perp \rangle \langle \ell_1^\perp q \rangle}{\langle \ell_2^\perp q \rangle} \frac{[2\ 1]}{\langle 1\ 2 \rangle \langle 1 | \ell_1 | 1 \rangle}, \\
\mathcal{A}_4(1^+, 2^+, \ell_2^{S_a}, \ell_1^{S_b}) &= -\mu^2 \frac{[2\ 1]}{\langle 1\ 2 \rangle \langle 1 | \ell_1 | 1 \rangle} \delta^{ab}, \tag{2.9}
\end{aligned}$$

We have checked \mathcal{A}_4 with the BCFW recursion relations. These expressions inspire us to guess the general expression for \mathcal{A}_n for $n \geq 4$. The notations here are the same as in the following subsection §2.4.

2.4 General expressions for the n-point amplitudes

To express n-point tree amplitudes, we denote k positive helicity gluons as $1, 2, \dots, k$, and 2 adjacent massless particle with extra-dimensional momenta by their 4-dimensional momenta ℓ_1, ℓ_2 . (In this paper, we also denote this series of amplitudes by $\{\ell_2^{h_2}, \ell_1^{h_1}\}$ for short)

Using the BCFW recursion relations, we find $n = k + 2$ tree amplitudes do have general expressions for $n \geq 4$. Here we list our main results

$$\begin{aligned}
\mathcal{A}_n(1^+, 2^+, \dots, k^+, \ell_2^+, \ell_1^+) &= \mathcal{A}_n(1^+, 2^+, \dots, k^+, \ell_2^+, \ell_1^{S_a}) = 0, \\
\mathcal{A}_n(1^+, 2^+, \dots, k^+, \ell_2^+, \ell_1^-) &= -\frac{\mu^2 \langle q \ell_1^\perp \rangle^2}{\langle q \ell_2^\perp \rangle^2} \times \frac{[1 | \prod_{i=2}^{k-1} (\mu^2 - x_{i, \ell_1} x_{\ell_1, i+1}) | k]}{\langle 1\ 2 \rangle \dots \langle k-1\ k \rangle \prod_{i=2}^k (\mu^2 + x_{\ell_1, i}^2)}, \\
\mathcal{A}_n(1^+, 2^+, \dots, k^+, \ell_2^-, \ell_1^-) &= \langle \ell_1^\perp \ell_2^\perp \rangle^2 \times \frac{[1 | \prod_{i=2}^{k-1} (\mu^2 - x_{i, \ell_1} x_{\ell_1, i+1}) | k]}{\langle 1\ 2 \rangle \dots \langle k-1\ k \rangle \prod_{i=2}^k (\mu^2 + x_{\ell_1, i}^2)}, \\
\mathcal{A}_n(1^+, 2^+, \dots, k^+, \ell_2^-, \ell_1^{S_a}) &= -\frac{\sqrt{2} \mu_{\ell_1, a} \langle \ell_1^\perp \ell_2^\perp \rangle \langle \ell_2^\perp q \rangle}{\langle \ell_1^\perp q \rangle} \times \frac{[1 | \prod_{i=2}^{k-1} (\mu^2 - x_{i, \ell_1} x_{\ell_1, i+1}) | k]}{\langle 1\ 2 \rangle \dots \langle k-1\ k \rangle \prod_{i=2}^k (\mu^2 + x_{\ell_1, i}^2)}, \\
\mathcal{A}_n(1^+, 2^+, \dots, k^+, \ell_2^{S_a}, \ell_1^-) &= \frac{\sqrt{2} \mu_{\ell_2, a} \langle \ell_2^\perp \ell_1^\perp \rangle \langle \ell_1^\perp q \rangle}{\langle \ell_2^\perp q \rangle} \times \frac{[1 | \prod_{i=2}^{k-1} (\mu^2 - x_{i, \ell_1} x_{\ell_1, i+1}) | k]}{\langle 1\ 2 \rangle \dots \langle k-1\ k \rangle \prod_{i=2}^k (\mu^2 + x_{\ell_1, i}^2)}, \\
\mathcal{A}_n(1^+, 2^+, \dots, k^+, \ell_2^{S_a}, \ell_1^{S_b}) &= \mu^2 \delta^{ab} \times \frac{[1 | \prod_{i=2}^{k-1} (\mu^2 - x_{i, \ell_1} x_{\ell_1, i+1}) | k]}{\langle 1\ 2 \rangle \dots \langle k-1\ k \rangle \prod_{i=2}^k (\mu^2 + x_{\ell_1, i}^2)}, \tag{2.10}
\end{aligned}$$

where q is a null 4-dimensional reference momentum and $q \cdot \ell_i \neq 0$, $\ell_i^\perp = \ell_i - \frac{\vec{\mu}^2}{\langle q | \ell_i | q \rangle} q$, μ^2 is the inner product of all the extra dimensional momentum values of two $\widehat{\ell}_i$, while $\mu_{\ell_i, a}$ is the extra-dimensional momentum component of $\widehat{\ell}_i$ in the direction \vec{e}_a of the extra-dimensional polarization vector $\epsilon^{S_a}(\ell_i)$. We borrow the notations $x_{i, \ell_1} = p_i + p_{i+1} + \dots + p_k + p_{\ell_2}$ and $x_{\ell_1, i} = p_{\ell_1} + p_1 + \dots + p_{i-1}$ from [15] and set $[1 | \prod_{i=2}^{k-1} (\mu^2 - x_{i, \ell_1} x_{\ell_1, i+1}) | 2] = [1 \ 2]$ for $n=4$ ($k=2$), so these expressions also hold for \mathcal{A}_4 . Massive versions of $\{\ell_2^{S_a}, \ell_1^{S_b}\}$ and $\{\ell_2^+, \ell_1^-\}$ have been obtained in [14] and [15] separately but the other two $\{\ell_2^-, \ell_1^-\}$ and $\{\ell_2^-, \ell_1^{S_a}\}$ are new here.

One observation is that these amplitudes have a common structure: they all have one common part $\frac{[1 | \prod_{i=2}^{k-1} (\mu^2 - x_{i, \ell_1} x_{\ell_1, i+1}) | k]}{\langle 1 \ 2 \rangle \dots \langle k-1 \ k \rangle \prod_{i=2}^k (\mu^2 + x_{\ell_1, i}^2)}$ changing with the increasing number of k , while the other part, stays the same for any number of k and depends on $\ell_1^{h_1}, \ell_2^{h_2}, q, \mu_{\ell_i, a}, \mu^2$. This structure is astonishingly simple!

2.5 Structure of the general expressions

The general expressions for n -points amplitudes with $n > 4$ are not easy to get from Feynman-diagrams. Here based on our assumption that they have a common structure inspired by 4-point cases (2.9), we build a clever proof by using the BCFW recursion relations.

Let's study the common structure first. We can see \mathcal{A}_n in (2.10) have two parts, we denote them as $PartA(\ell_1^{h_1}, \ell_2^{h_2})$ and $PartB(k)$.

$PartB(k)$ is same for all non-zero cases:

$$PartB(k) = \frac{[1 | \prod_{i=2}^{k-1} (\mu^2 - x_{i, \ell_1} x_{\ell_1, i+1}) | k]}{\langle 1 \ 2 \rangle \dots \langle k-1 \ k \rangle \prod_{i=2}^k (\mu^2 + x_{\ell_1, i}^2)} \quad (2.11)$$

While $PartA$ is different

$$\begin{aligned} PartA(\ell_1^+, \ell_2^-) &= -\frac{\mu^2 \langle q \ell_1^\perp \rangle^2}{\langle q \ell_2^\perp \rangle^2}, \\ PartA(\ell_1^-, \ell_2^-) &= \langle \ell_1^\perp \ell_2^\perp \rangle^2, \\ PartA(\ell_1^-, \ell_2^{S_a}) &= -\frac{\sqrt{2} \mu_{\ell_1, a} \langle \ell_1^\perp \ell_2^\perp \rangle \langle \ell_2^\perp q \rangle}{\langle \ell_1^\perp q \rangle}, \\ PartA(\ell_1^{S_a}, \ell_2^-) &= \frac{\sqrt{2} \mu_{\ell_2, a} \langle \ell_2^\perp \ell_1^\perp \rangle \langle \ell_1^\perp q \rangle}{\langle \ell_2^\perp q \rangle}, \\ PartA(\ell_1^{S_a}, \ell_2^{S_b}) &= \mu^2 \delta_{ab}, \end{aligned} \quad (2.12)$$

so $PartA(\ell_2^h, \ell_1^{h_1})$ is independent of all k gluons.

The pole structure of these \mathcal{A}_n is contained in $PartB(k)$. We see there are not only 2-particle poles $\langle 1 \ 2 \rangle \dots \langle k-1 \ k \rangle$ in $PartB(k)$ but also multi-particle poles $\prod_{i=2}^k (\mu^2 + x_{\ell_1, i}^2)$.

3. Proof by Mathematical Induction

In this section, we give our proof by mathematical induction of the results (2.10). First we choose a special shift so that the BCFW recursion relations for these tree amplitudes can be reduced much simple. Then we separate our proof for the inductive step into two parts.

As the first step of a mathematical induction, known as the base case, we can see \mathcal{A}_4 amplitudes given in (2.9) match the expressions (2.10) for $n = 4$.

The second step is called the inductive step. let's assume the expressions (2.10) hold for n -point cases. If the expressions (2.10) hold for $(n+1)$ -point cases, we can conclude that (2.10) are general expressions for $n \geq 4$. This hypothesis the expressions (2.10) hold for $(n+1)$ -point cases, is called the inductive hypothesis. By prove the hypothesis, we can prove the expressions hold for any natural number n for $n \geq 4$.

3.1 Strategies to use the BCFW recursion relations

Using the BCFW recursion relations [5, 6], higher-point tree amplitudes can be obtained from lower-point tree amplitudes by

$$A_{higher} = \sum_{\text{all partition}} \frac{A_L(z = z_L) A_R(z = z_L)}{P(z = 0)^2} \quad (3.1)$$

An instruction to use the BCFW recursion relations and notations can be found in textbooks such as [17]. In our cases, the intermedia particle have extra-dimensional momentum, so we need to sum over the intermedia particle's helicity $+-S_a$ rather than $+-$ in 4-D cases.

To calculate \mathcal{A}_{n+1} ($n=k+2$) from lower-point amplitudes, we choose the BCFW shift to be $[k|k+1]^+$

$$|\widehat{k}] = |k] - z|k+1], \quad |\widehat{k}\rangle = |k\rangle, \quad |\widehat{k+1}] = |k+1], \quad |\widehat{k+1}\rangle = |k+1\rangle + z|k\rangle. \quad (3.2)$$

This shift involving two adjacent gluons mainly have two advantages. One is that poles $\langle 1 2 \rangle \cdots \langle k-1 k \rangle$ and $\prod_{i=2}^k (\mu^2 + x_{\ell_1, i}^2)$ in the denominator of $PartB(k)$ would not change after the shift.

The other advantage is that we can reduce (3.1) to

$$\mathcal{A}_{n+1}(1^+, 2^+, \dots, (k+1)^+, \ell_2^{h_2}, \ell_1^{h_1}) = \sum_h \frac{A_L(1^+, 2^+, \dots, \widehat{k}^+, m\widehat{Q}^h, \ell_1^{h_1}) \times A_R(\widehat{k+1}^+, \ell_2^{h_2}, \widehat{Q}^{-h})}{P(0)^2}, \quad (3.3)$$

where $P_{m\widehat{Q}} = -P_{\widehat{Q}} = P_{\widehat{k+1}} + P_{\ell_2}$ for momentum conservation. Symbol $\widehat{}$ in (3.3) stands for that the leg is on-shell after the shift. We see one only need to replace ℓ_2 with $m\widehat{Q}$ in A_n to express A_L , which is the other advantage we choose $[k|k+1]^+$

The reason why we can obtain (3.3) is as following. When both of ℓ_1 and ℓ_2 are in the same diagram A_L or A_R , the intermedia massless particles denoted as \widehat{Q} and $m\widehat{Q}$ will have no extra-dimensional momentum component. One can find the other diagram which have no leg with extra-dimensional momentum always vanishes. In a conclusion, a non-vanishing $A_L \times A_R$ term should be a product of a n -point amplitude and a 3-point amplitude as (3.3).

For zero cases in (2.10), using (3.3) we find are they do vanish because Right Hand Side of (3.3) of $\{\ell_2^+, \ell_1^+\}, \{\ell_2^+, \ell_1^{S_a}\}, \{\ell_2^{S_a}, \ell_1^+\}$ have at least one vanishing lower-point amplitude of A_L and A_R .

Finally, let's focus on non-zero cases. Our strategies for the proof is to deal with *PartA* and *PartB* separately in our recursion from n to $n + 1$. We rewrite (3.3) as

$$\begin{aligned} \mathcal{A}_{n+1}(1^+, 2^+, \dots, (k+1)^+, \ell_2^{h_2}, \ell_1^{h_1}) &= \sum_h \frac{A_L(1^+, 2^+, \dots, \widehat{k}^+, m\widehat{Q}^h, \ell_1^{h_1}) \times A_R(\widehat{k+1}^+, \ell_2^{h_2}, \widehat{Q}^{-h})}{P(0)^2} \\ &= \frac{PartB(\widehat{k}) \sum_h PartA(m\widehat{Q}^h, \ell_1^{h_1}) \times A_3(\widehat{k+1}^+, \widehat{Q}^{-h}, \ell_2^{h_2})}{P(0)^2} \end{aligned} \quad (3.4)$$

With these notations, we separate the proof to two parts. The first part is

$$\sum_h PartA(m\widehat{Q}^h, \ell_1^{h_1}) \times A_3(\widehat{k+1}^+, \widehat{Q}^{-h}, \ell_2^{h_2}) \rightarrow PartA(\ell_2^h, \ell_1^{h_1}) \frac{\langle k | \ell_2 | k+1 \rangle}{\langle k | k+1 \rangle} \quad (3.5)$$

The second part is:

$$\begin{aligned} PartB(\widehat{k}) &= \frac{[1 | \prod_{i=2}^{k-1} (\mu^2 - x_{i,\ell_1} x_{\ell_1,i+1}) | \widehat{k}]}{\langle 1 | 2 \rangle \dots \langle k-1 | k \rangle \prod_{i=2}^k (\mu^2 + x_{\ell_1,i}^2)} \rightarrow \frac{[1 | \prod_{i=2}^k (\mu^2 - x_{i,\ell_1} x_{\ell_1,i+1}) | k+1]}{\langle 1 | 2 \rangle \dots \langle k-1 | k \rangle \prod_{i=2}^k (\mu^2 + x_{\ell_1,i}^2)} \frac{1}{\langle k | \ell_2 | k+1 \rangle} \\ &= PartB(k+1) \frac{\langle k | k+1 \rangle P(0)^2}{\langle k | \ell_2 | k+1 \rangle} \end{aligned} \quad (3.6)$$

According to the partition in (3.3), where two legs $(k+1)$ and ℓ_2 are always on the right side, we can rewrite $P(0)^2$

$$P(0)^2 = (p_{\widehat{\ell_2}} + p_{k+1})^2 = (p_{\ell_1} + p_1 + p_2 + \dots + p_k)^2 + \mu^2 \quad (3.7)$$

Define $x_{\ell_1,i} = p_{\ell_1} + p_1 + p_2 + \dots + p_{i-1}$, then

$$P(0)^2 = \mu^2 + x_{\ell_1,k}^2 \quad (3.8)$$

The on-shell condition for $P(z)^2 = 0$ is $P_{\widehat{Q}}^2 = (P_{\widehat{k+1}} + P_{\widehat{\ell_2}})^2 = 0$, with the solution for z

$$z = z_L(k+1) = -\frac{\langle k+1 | \ell_2 | k+1 \rangle}{\langle k | \ell_2 | k+1 \rangle} \quad (3.9)$$

3.2 Proof of the inductive hypothesis

Here we prove the inductive hypothesis by showing that (3.5) and (3.6) are correct.

After the shift, $PartB(k)$ changes with $[\widehat{k}]$

$$[\widehat{k}] = [k] - z_L(k+1)[k+1] = [k] + \frac{\langle k+1 | \ell_2 | k+1 \rangle}{\langle k | \ell_2 | k+1 \rangle} [k+1] = \frac{\mu^2 - x_{k,\ell_1} x_{\ell_1,k+1}}{\langle k | \ell_2 | k+1 \rangle} [k+1] \quad (3.10)$$

where we use identity $|k|\ell_2|k+1| + |k+1|\ell_2|k+1| = |\mu^2|k+1| - |(p_k + p_{k+1} + p_{\ell_2})|\ell_1 + 1 + \dots + k|k+1|$

Applying (3.10) to $PartB(k)$, we see (3.6) is true. This is also an advantage of the BCFW shift we choose.

The other part of the inductive step (3.5) is different for each non-zero cases.

The remaining non-zero cases are $\{\ell_2^+, \ell_1^-\}$, $\{\ell_2^-, \ell_1^+\}$, $\{\ell_2^-, \ell_1^{S_a}\}$, $\{\ell_2^{S_a}, \ell_1^-\}$, $\{\ell_2^-, \ell_1^-\}$, $\{\ell_2^{S_a}, \ell_1^{S_b}\}$. We only need to calculate 4 of them $\{\ell_2^+, \ell_1^-\}$, $\{\ell_2^-, \ell_1^{S_a}\}$, $\{\ell_2^{S_a}, \ell_1^{S_b}\}$, $\{\ell_2^-, \ell_1^-\}$. Regarding the results for $\{\ell_2^+, \ell_1^+\}$, $\{\ell_2^+, \ell_1^{S_a}\}$, $\{\ell_2^{S_a}, \ell_1^+\}$ cases are zero, we can reduce the number of contributing terms for non-zero cases.

Recall (3.5) is to show

$$\sum_h PartA(m\widehat{Q}^h, \ell_1^{h_1}) \times A_3(\widehat{k+1}^+, \widehat{Q}^{-h}, \ell_2^{h_2}) = \frac{PartA(\ell_2^{h_2}, \ell_1^{h_1}) \times \langle k|\ell_2|k+1 \rangle}{\langle k, k+1 \rangle} \quad (3.11)$$

We prove (3.11) by the following steps.

First, substitute all the spinor brackets and angles of Q^\perp with associated spinor brackets and angles of $m\widehat{Q}^\perp$ due to the momentum conservation. Second, multiply every $|m\widehat{Q}^\perp]$ (or $|m\widehat{Q}^\perp\rangle$) with $\langle m\widehat{Q}^\perp q \rangle$ (or $[m\widehat{Q}^\perp q]$) so that we can replace $m\widehat{Q}^\perp$ by $m\widehat{Q}$ for $m\widehat{Q}^\perp = m\widehat{Q} - \frac{\vec{\mu}^2}{\langle q|m\widehat{Q}|q \rangle}q$.

Next before simplifying the spinor products, if there is $\langle q|m\widehat{Q}|q \rangle$ appearing in the denominator, we should eliminate all the $\langle q|m\widehat{Q}|q \rangle$ in the first place, by using the Schouten Identity to the $\langle \ell_1^\perp|m\widehat{Q}|q \rangle$ with another term. If there is no $\langle q|m\widehat{Q}|q \rangle$ left, we begin to simplify the spinor product using the following identities

$$\begin{aligned} \langle q|\widehat{k+1} \rangle &= \frac{-\langle k|k+1 \rangle [k+1|\ell_2^\perp] \langle \ell_2^\perp q \rangle}{\langle k|\ell_2|k+1 \rangle} \\ \langle \widehat{k+1}|\ell_2^\perp \rangle &= \frac{\mu^2 \langle k|k+1 \rangle [k+1|q]}{[\ell_2^\perp q] \langle k|\ell_2|k+1 \rangle} \\ \langle q|m\widehat{Q}|\ell_2^\perp \rangle &= \frac{\langle \ell_2^\perp|q \rangle \langle k|k+1 \rangle [k+1|\ell_2^\perp]^2}{\langle k|\ell|k+1 \rangle} \\ \langle \ell_2^\perp|m\widehat{Q}|q \rangle &= -\frac{\mu^2 \langle k|k+1 \rangle [q|k+1]^2}{\langle k|\ell_2|k+1 \rangle [q|\ell_2^\perp]} \\ \langle q|m\widehat{Q}|k+1 \rangle &= \langle q|\ell_2^\perp|k+1 \rangle \end{aligned} \quad (3.12)$$

which is no hard to prove by substituting $\widehat{k+1}$, \widehat{k} and $m\widehat{Q}$ then using the Schouten identity to combine the separating terms.

In the following section, we will prove (3.11) is true for each non-zero cases in detail. We have also verified (3.11) numerically using S@M package.[18].

3.2.1 $\{\ell_2^+, \ell_1^-\}$ and $\{\ell_2^{S_a}, \ell_1^{S_b}\}$

Both of the two cases $\{\ell_2^+, \ell_1^-\}$ and $\{\ell_2^{S_a}, \ell_1^{S_b}\}$ only have one contributing term and no $\langle q|m\widehat{Q}|q \rangle$ will appear in the denominators. We only need to simplify all the spinor products.

For $\{\ell_2^+, \ell_1^-\}$

$$\begin{aligned}
& PartA(m\widehat{Q}^+, \ell_1^-) \times A_3(\widehat{k+1}, \widehat{Q}^-, \ell_2^+) \\
&= -\frac{u^2 \langle q \ell_1^\perp \rangle^2}{\langle q \widehat{Q}^\perp \rangle^2} \times \frac{[k+1 \ell_2^\perp]^3}{[k+1 m\widehat{Q}^\perp][m\widehat{Q}^\perp \ell_2^\perp]} \\
&= -\frac{u^2 \langle q \ell_1^\perp \rangle^2 [k+1 \ell_2^\perp]^3}{\langle q|m\widehat{Q}|k+1 \rangle \langle q|m\widehat{Q}|\ell_2^\perp \rangle} \\
&= -\frac{\mu^2 \langle q \ell_1^\perp \rangle^2}{\langle q \ell_2^\perp \rangle^2} \times \frac{\langle k|\ell_2|k+1 \rangle}{\langle k k+1 \rangle}
\end{aligned} \tag{3.13}$$

In the last line, we using the identities in (3.12).

The case $\{\ell_2^{S_a}, \ell_1^{S_b}\}$ is similar to $\{\ell_2^+, \ell_1^-\}$, we can easily get

$$\begin{aligned}
& \sum_{S_c} PartA(m\widehat{Q}^{S_c}, \ell_1^{S_b}) \times A_3(\widehat{k+1}^+, \widehat{Q}^{S_c}, \ell_2^{S_a}) \\
&= \sum_{S_c} -\mu^2 \delta_{cb} \times (-\delta_{ac} \frac{[k+1 \ell_2^\perp][k+1 \widehat{Q}^\perp]}{[\widehat{Q}^\perp \ell_2^\perp]}) \\
&= -\mu^2 \delta_{ab} \frac{[k+1 \ell_2^\perp][k+1|m\widehat{Q}^\perp|q]}{\langle q|m\widehat{Q}^\perp|\ell_2^\perp \rangle} \\
&= -\mu^2 \delta_{ab} \times \frac{\langle k|\ell_2|k+1 \rangle}{\langle k k+1 \rangle}
\end{aligned} \tag{3.14}$$

3.2.2 $\{\ell_2^-, \ell_1^{S_b}\}$

This case has two contributing terms

$$\begin{aligned}
& \sum_h PartA(m\widehat{Q}^h, \ell_1^{h_1}) \times A_R(\widehat{k+1}^+, \ell_2^{h_2}, \widehat{Q}^{-h}) = PartA(m\widehat{Q}^-, \ell_1^{S_a}) \times \mathcal{A}_3(\widehat{k+1}^+, \ell_2^-, \widehat{Q}^+) \\
& \quad + \sum_{S_b} PartA(m\widehat{Q}^{S_b}, \ell_1^{S_a}) \times \mathcal{A}_3(\widehat{k+1}^+, \ell_2^-, \widehat{Q}^{S_b}),
\end{aligned} \tag{3.15}$$

First we need to separate $\mathcal{A}_3(1^+, \ell_2^-, \ell_1^{S_a})$ to two terms

$$\mathcal{A}_3(1^+, \ell_2^-, \ell_1^{S_a}) = \sqrt{2} \mu_{\ell_1, a} \frac{\langle \ell_2^\perp q \rangle [1 q]}{\langle q 1 \rangle [q \ell_2^\perp]} + \sqrt{2} \mu_{\ell_1, a} \frac{\langle \ell_2^\perp q \rangle^2 [1 q]}{\langle q 1 \rangle \langle \ell_1^\perp q \rangle [q \ell_1^\perp]} \tag{3.16}$$

Then we write them explicitly

$$\begin{aligned}
PartA(m\widehat{Q}^-, \ell_1^{S_a}) \times \mathcal{A}_3(\widehat{k+1}^+, \ell_2^-, \widehat{Q}^+) &= \frac{-\sqrt{2} \mu_{\ell_1, a} \langle \ell_1^\perp m\widehat{Q}^\perp \rangle \langle m\widehat{Q}^\perp q \rangle [m\widehat{Q}^\perp k+1]^3}{\langle \ell_1^\perp q \rangle [k+1 \ell_2^\perp] [\ell_2^\perp m\widehat{Q}^\perp]} \\
&= \frac{-\sqrt{2} \mu_{\ell_1, a} \langle \ell_1^\perp |m\widehat{Q}^\perp|q \rangle \langle q|m\widehat{Q}^\perp|k+1 \rangle^3}{\langle \ell_1^\perp q \rangle [k+1 \ell_2^\perp] \langle q|m\widehat{Q}^\perp|\ell_2^\perp \rangle \langle q|m\widehat{Q}^\perp|q \rangle}
\end{aligned} \tag{3.17}$$

and

$$\begin{aligned}
& \sum_{S_b} \text{Part}A(m\widehat{Q}^{S_b}, \ell_1^{S_a}) \times \mathcal{A}_3(\widehat{k+1}^+, \ell_2^-, \widehat{Q}^{S_b}) \\
&= \sum_{S_b} (\mu^2 \delta_{ab} \sqrt{2} \mu_{\ell_1, b} \frac{\langle \ell_2^\perp q \rangle [k+1 q]}{\langle q \widehat{k+1} \rangle [q \ell_2^\perp]} + \mu^2 \delta_{ab} \sqrt{2} \mu_{\ell_1, b} \frac{\langle \ell_2^\perp q \rangle^2 [k+1 q]}{\langle q \widehat{k+1} \rangle \langle m\widehat{Q}^\perp q \rangle [q m\widehat{Q}^\perp]}) \\
&= \mu^2 \sqrt{2} \mu_{\ell_1, a} \frac{\langle \ell_2^\perp q \rangle [k+1 q]}{\langle q \widehat{k+1} \rangle [q \ell_2^\perp]} - \mu^2 \sqrt{2} \mu_{\ell_1, a} \frac{\langle \ell_2^\perp q \rangle^2 [k+1 q]}{\langle q \widehat{k+1} \rangle \langle q | m\widehat{Q} | q \rangle} \tag{3.18}
\end{aligned}$$

Instead of simplifying all the spinor products directly, the first thing to do is to apply the Schouten Identity to $\langle \ell_1^\perp | m\widehat{Q} | q \rangle \langle q | m\widehat{Q} | k+1 \rangle$ to eliminate the unphysical pole $\langle q | m\widehat{Q} | q \rangle$.

$$\langle \ell_1^\perp | m\widehat{Q} | q \rangle \langle q | m\widehat{Q} | k+1 \rangle = \langle \ell_1^\perp | m\widehat{Q} | k+1 \rangle \langle q | m\widehat{Q} | q \rangle + \mu^2 \langle \ell_1^\perp q \rangle [k+1 q] \tag{3.19}$$

In identity (3.19), $\langle \ell_1^\perp | m\widehat{Q} | k+1 \rangle$ can be write as

$$\begin{aligned}
\langle \ell_1^\perp | m\widehat{Q} | k+1 \rangle &= \langle \ell_1^\perp | \ell_2 | k+1 \rangle \\
&= \langle \ell_1^\perp | \ell_2^\perp + \frac{\mu^2}{\langle q | \ell_2 | q \rangle} q | k+1 \rangle \\
&= \langle \ell_1^\perp \ell_2^\perp \rangle [\ell_2^\perp k+1] + \frac{\mu^2 \langle \ell_1^\perp q \rangle [q k+1]}{\langle q \ell_2^\perp \rangle [\ell_2^\perp q]} \tag{3.20}
\end{aligned}$$

Then the following can be merged and simplified

$$\begin{aligned}
& \sum_h \text{Part}A(m\widehat{Q}^h, \ell_1^{S_a}) \times A_3(\widehat{k+1}, \widehat{Q}^{-h}, \ell_2^-) \\
&= \text{Part}A(m\widehat{Q}^-, \ell_1^{S_a}) \times A_3(\widehat{k+1}, \widehat{Q}^+, \ell_2^-) + \text{Part}A(m\widehat{Q}^{S_a}, \ell_1^{S_a}) \times A_3(\widehat{k+1}, \widehat{Q}^{S_a}, \ell_2^-) \\
&= \frac{\sqrt{2} \mu_{\ell_1, a} \langle k | \ell_2 | k+1 \rangle \langle \ell_1^\perp | m\widehat{Q}^\perp | q \rangle \langle q | m\widehat{Q}^\perp | k+1 \rangle}{\langle \ell_1^\perp q \rangle [k+1 \ell_2^\perp] \langle k k+1 \rangle \langle q | m\widehat{Q}^\perp | q \rangle} - \frac{\mu^2 \sqrt{2} \mu_{\ell_1, a} [k+1 q] \langle k | \ell_2 | k+1 \rangle}{\langle k k+1 \rangle [k+1 \ell_2^\perp] [\ell_2^\perp q]} \\
&+ \frac{\mu^2 \sqrt{2} \mu_{\ell_1, a} \langle \ell_2^\perp q \rangle [k+1 q] \langle k | \ell_2 | k+1 \rangle}{\langle k k+1 \rangle [k+1 \ell_2^\perp]} \\
&= \frac{\sqrt{2} \mu_{\ell_1, a} \langle \ell_2^\perp \ell_1^\perp \rangle \langle \ell_1^\perp q \rangle \langle k | \ell_2 | k+1 \rangle}{\langle \ell_2^\perp q \rangle \langle k k+1 \rangle} \tag{3.21}
\end{aligned}$$

3.2.3 $\{l_2^-, l_1^-\}$

$\{l_2^-, l_1^-\}$ case has three contributing terms

$$\begin{aligned}
\sum_h PartA(m\hat{Q}^h, \ell_1^{h_1}) \times A_R(\widehat{k+1}^+, \ell_2^{h_2}, m\hat{Q}^{-h}) &= PartA(m\hat{Q}^-, \ell_1^-) \times \mathcal{A}_3(\widehat{k+1}^+, \ell_2^-, \hat{Q}^+) \\
&+ PartA(m\hat{Q}^+, \ell_1^-) \times \mathcal{A}_3(\widehat{k+1}^+, \ell_2^-, \hat{Q}^-) \\
&+ \sum_{Sc} PartA(m\hat{Q}^{Sc}, \ell_1^-) \times \mathcal{A}_3(\widehat{k+1}^+, \ell_2^-, \hat{Q}^{Sc}) ,
\end{aligned} \tag{3.22}$$

We write them explicitly

$$\begin{aligned}
PartA(m\hat{Q}^-, \ell_1^-) \times \mathcal{A}_3(\widehat{k+1}^+, \ell_2^-, \hat{Q}^+) &= \frac{\langle m\hat{Q}^\perp \ell_1^\perp \rangle^2 [m\hat{Q}^\perp \widehat{k+1}]}{[\widehat{k+1} \ell_2^\perp] [\ell_2^\perp m\hat{Q}^\perp]} \\
&= \frac{-\langle \ell_1^\perp | m\hat{Q}^\perp | q \rangle^2 \langle q | m\hat{Q}^\perp | \widehat{k+1} \rangle^3}{[\widehat{k+1} \ell_2^\perp] \langle q | m\hat{Q}^\perp | q \rangle^2 \langle q | m\hat{Q}^\perp | \ell_2^\perp \rangle} , \\
PartA(m\hat{Q}^+, \ell_1^-) \times \mathcal{A}_3(\widehat{k+1}^+, \ell_2^-, \hat{Q}^-) &= \frac{-\mu^2 \langle \ell_1^\perp q \rangle^2 \langle \ell_2^\perp m\hat{Q}^\perp \rangle^3}{\langle \widehat{k+1} m\hat{Q}^\perp \rangle \langle \widehat{k+1} \ell_2^\perp \rangle \langle q m\hat{Q}^\perp \rangle^2} \\
&= \frac{-\mu^2 \langle \ell_1^\perp q \rangle^2 \langle \ell_2^\perp | m\hat{Q}^\perp | q \rangle^3}{\langle \widehat{k+1} m\hat{Q}^\perp \rangle \langle q | m\hat{Q}^\perp | q \rangle^2 \langle \widehat{k+1} | m\hat{Q}^\perp | q \rangle} , \\
\sum_{Sc} PartA(m\hat{Q}^{Sc}, \ell_1^-) \times \mathcal{A}_3(\widehat{k+1}^+, \ell_2^-, \hat{Q}^{Sc}) &= \frac{2\mu^2 \langle m\hat{Q}^\perp \ell_1^\perp \rangle \langle \ell_1^\perp q \rangle \langle \ell_2^\perp q \rangle \langle \ell_2^\perp m\hat{Q}^\perp \rangle^2}{\langle \widehat{k+1} \ell_2^\perp \rangle \langle \widehat{k+1} m\hat{Q}^\perp \rangle \langle m\hat{Q}^\perp q \rangle^2} \\
&= \frac{2\mu^2 \langle \ell_1^\perp q \rangle \langle \ell_2^\perp q \rangle \langle \ell_1^\perp | m\hat{Q}^\perp | q \rangle \langle \ell_2^\perp | m\hat{Q}^\perp | q \rangle^2}{\langle \widehat{k+1} \ell_2^\perp \rangle \langle q | m\hat{Q}^\perp | q \rangle^2 \langle \widehat{k+1} | m\hat{Q}^\perp | q \rangle} ,
\end{aligned} \tag{3.23}$$

So

$$\begin{aligned}
& \sum_h \text{Part} A(m\widehat{Q}^h, \ell_1^-) \times A_R(\widehat{k+1}^+, \ell_2^-, m\widehat{Q}^{-h}) \\
&= \frac{-\langle \ell_1^\perp | m\widehat{Q}^\perp | q \rangle^2 \langle q | m\widehat{Q}^\perp | \widehat{k+1} \rangle^3}{[\widehat{k+1} \ell_2^\perp] \langle q | m\widehat{Q}^\perp | q \rangle^2 \langle q | m\widehat{Q}^\perp | \ell_2^\perp \rangle} + \frac{-\mu^2 \langle \ell_1^\perp q \rangle^2 \langle \ell_2^\perp | m\widehat{Q}^\perp | q \rangle^3}{\langle \widehat{k+1} m\widehat{Q}^\perp \rangle \langle q | m\widehat{Q}^\perp | q \rangle^2 \langle \widehat{k+1} | m\widehat{Q}^\perp | q \rangle} \\
&+ \frac{\mu^2 \langle \ell_1^\perp q \rangle \langle \ell_2^\perp q \rangle \langle \ell_1^\perp | m\widehat{Q}^\perp | q \rangle \langle \ell_2^\perp | m\widehat{Q}^\perp | q \rangle^2}{\langle \widehat{k+1} \ell_2^\perp \rangle \langle q | m\widehat{Q}^\perp | q \rangle^2 \langle \widehat{k+1} | m\widehat{Q}^\perp | q \rangle} + \frac{\mu^2 \langle \ell_1^\perp q \rangle \langle \ell_2^\perp q \rangle \langle \ell_1^\perp | m\widehat{Q}^\perp | q \rangle \langle \ell_2^\perp | m\widehat{Q}^\perp | q \rangle^2}{\langle \widehat{k+1} \ell_2^\perp \rangle \langle q | m\widehat{Q}^\perp | q \rangle^2 \langle \widehat{k+1} | m\widehat{Q}^\perp | q \rangle} \\
&= \frac{-\mu^2 \langle \ell_2^\perp q \rangle \langle \ell_1^\perp | m\widehat{Q}^\perp | q \rangle^2 \langle \ell_2^\perp | m\widehat{Q}^\perp | q \rangle}{\langle \widehat{k+1} \ell_2^\perp \rangle \langle q | m\widehat{Q}^\perp | q \rangle^2 \langle \widehat{k+1} | m\widehat{Q}^\perp | q \rangle} + \frac{\mu^2 \langle \ell_1^\perp \ell_2^\perp \rangle \langle \ell_2^\perp q \rangle \langle \ell_1^\perp | m\widehat{Q}^\perp | q \rangle \langle \ell_2^\perp | m\widehat{Q}^\perp | q \rangle}{\langle \widehat{k+1} \ell_2^\perp \rangle \langle q | m\widehat{Q}^\perp | q \rangle \langle \widehat{k+1} | m\widehat{Q}^\perp | q \rangle} \\
&+ \frac{\mu^2 \langle \ell_1^\perp \ell_2^\perp \rangle \langle \ell_1^\perp q \rangle \langle \ell_2^\perp | m\widehat{Q}^\perp | q \rangle^2}{\langle \widehat{k+1} \ell_2^\perp \rangle \langle q | m\widehat{Q}^\perp | q \rangle \langle \widehat{k+1} | m\widehat{Q}^\perp | q \rangle} + \frac{\langle \ell_2^\perp q \rangle^3 \langle \ell_1^\perp | m\widehat{Q}^\perp | q \rangle^2 [\ell_2^\perp k+1]^2}{\langle q | m\widehat{Q}^\perp | q \rangle^2 \langle q | m\widehat{Q}^\perp | \ell_2^\perp \rangle} \\
&= \frac{1}{\langle q | m\widehat{Q}^\perp | q \rangle^2} \left(\frac{-\mu^2 \langle \ell_2^\perp q \rangle \langle \ell_1^\perp | m\widehat{Q}^\perp | q \rangle^2 \langle \ell_2^\perp | m\widehat{Q}^\perp | q \rangle}{\langle \widehat{k+1} \ell_2^\perp \rangle \langle \widehat{k+1} | m\widehat{Q}^\perp | q \rangle} - \frac{\langle \ell_2^\perp q \rangle^3 \langle \ell_1^\perp | m\widehat{Q}^\perp | q \rangle^2 [\ell_2^\perp k+1]^2}{\langle q | m\widehat{Q}^\perp | \ell_2^\perp \rangle} \right) \\
&+ \frac{1}{\langle q | m\widehat{Q}^\perp | q \rangle} \left(\frac{\mu^2 \langle \ell_1^\perp \ell_2^\perp \rangle \langle \ell_2^\perp q \rangle \langle \ell_1^\perp | m\widehat{Q}^\perp | q \rangle \langle \ell_2^\perp | m\widehat{Q}^\perp | q \rangle}{\langle \widehat{k+1} \ell_2^\perp \rangle \langle \widehat{k+1} | m\widehat{Q}^\perp | q \rangle} + \frac{\mu^2 \langle \ell_1^\perp \ell_2^\perp \rangle \langle \ell_1^\perp q \rangle \langle \ell_2^\perp | m\widehat{Q}^\perp | q \rangle^2}{\langle \widehat{k+1} \ell_2^\perp \rangle \langle \widehat{k+1} | m\widehat{Q}^\perp | q \rangle} \right) \\
&= \frac{-\mu^2 \langle \ell_1^\perp \ell_2^\perp \rangle^2 \langle \ell_2^\perp | m\widehat{Q}^\perp | q \rangle}{\langle \widehat{k+1} \ell_2^\perp \rangle \langle \widehat{k+1} | m\widehat{Q}^\perp | q \rangle} \\
&= \frac{\langle \ell_1^\perp \ell_2^\perp \rangle^2 \langle k | \ell_2 | k+1 \rangle}{\langle k k+1 \rangle} \tag{3.24}
\end{aligned}$$

For the first equality we substitute (3.23) into (3.22). To get the second equality, we use the Schouten identity to factorize the last two terms. Here we should attention that one term use the first one, the other use the second one

$$\begin{aligned}
\langle \ell_1^\perp q \rangle \langle \ell_2^\perp | m\widehat{Q}^\perp | q \rangle &= \langle \ell_1^\perp \ell_2^\perp \rangle \langle q | m\widehat{Q}^\perp | q \rangle + \langle \ell_1^\perp | m\widehat{Q}^\perp | q \rangle \langle \ell_2^\perp q \rangle, \\
\langle \ell_2^\perp q \rangle \langle \ell_1^\perp | m\widehat{Q}^\perp | q \rangle &= \langle \ell_2^\perp \ell_1^\perp \rangle \langle q | m\widehat{Q}^\perp | q \rangle + \langle \ell_2^\perp | m\widehat{Q}^\perp | q \rangle \langle \ell_1^\perp q \rangle \tag{3.25}
\end{aligned}$$

And for the third equality, we combine terms according to their power of $\langle q | m\widehat{Q} | q \rangle$. To get the fourth equality, we should use the Schouten identity in (3.25) and equalities in (3.12).

4. All-plus 1-loop integrand in \mathcal{Q} -cut representation

After we get general expressions for these tree amplitudes, we can use them to calculate the loop integrand for color-ordered n-point gluon amplitude $\mathcal{A}_n^{1\text{-loop}}(1^+, 2^+, \dots, n^+)$ by \mathcal{Q} -cut construction.

The content below just follows the calculation for $A_4^{1\text{-loop}}(1^+, 2^+, 3^+, 4^+)$ in [2]. In the following steps, we find that the product $\mathcal{A}_L \mathcal{A}_R$ are independent of q , thus the loop integrand is also independent of q , which serves as a consistency check for our tree amplitudes.

The \mathcal{Q} -cut representation of the loop integrand is

$$\mathcal{I}_n^{\mathcal{Q}}(\ell) = \sum_{P_L} \sum_{h_1, h_2} \mathcal{A}_L(\cdots, \widehat{\ell}_R^{h_1}, -\widehat{\ell}_L^{h_2}) \frac{1}{\widetilde{\ell}^2 (-2\widetilde{\ell} \cdot P_L + P_L^2)} \mathcal{A}_R(\widehat{\ell}_L^{\bar{h}_2}, -\widehat{\ell}_R^{\bar{h}_1}, \cdots), \quad (4.1)$$

where $\widehat{\ell}_L = \alpha_L(\widetilde{\ell} + \vec{\eta})$, $\widehat{\ell}_R \equiv \widehat{\ell} - P_L$ with $\alpha_L = P_L^2/(2\widetilde{\ell} \cdot P_L) \neq 0$ and $\vec{\eta}^2 = \widetilde{\ell}^2$.

According to (4.1), the \mathcal{Q} -cut representation of the n -point all-plus 1-loop integrand is given by

$$\begin{aligned} \mathcal{I}^{\mathcal{Q}}(\widetilde{\ell}) &= \sum_{k=2}^{n-2} \sum_{h_1, h_2} \mathcal{A}_L(1^+, 2^+, \dots, k^+, \widehat{\ell}_R^{h_1}, -\widehat{\ell}_L^{h_2}) \frac{1}{\widetilde{\ell}^2 (-2\widetilde{\ell} \cdot p_L + p_L^2)} \mathcal{A}_R(\widehat{\ell}_L^{\bar{h}_2}, -\widehat{\ell}_R^{\bar{h}_1}, (k+1)^+, \dots, n^+) \\ &\quad + \text{Cyclic}\{p_1, p_2, \dots, p_{n-1}, p_n\} \\ &= \sum_{k=2}^{n-2} (\mathcal{A}_L(1^+, 2^+, \dots, k^+, \widehat{\ell}_R^+, -\widehat{\ell}_L^-) \mathcal{A}_R(\widehat{\ell}_L^-, -\widehat{\ell}_R^+, (k+1)^+, \dots, n^+) \\ &\quad + \mathcal{A}_L(1^+, 2^+, \dots, k^+, \widehat{\ell}_R^-, -\widehat{\ell}_L^+) \mathcal{A}_R(\widehat{\ell}_L^+, -\widehat{\ell}_R^-, (k+1)^+, \dots, n^+) \\ &\quad + \sum_{S_A} \mathcal{A}_L(1^+, 2^+, \dots, k^+, \widehat{\ell}_R^{S_A}, -\widehat{\ell}_L^{S_A}) \mathcal{A}_R(\widehat{\ell}_L^{S_A}, -\widehat{\ell}_R^{S_A}, (k+1)^+, \dots, n^+)) \frac{1}{\widetilde{\ell}^2 (-2\widetilde{\ell} \cdot p_L + p_L^2)} \\ &\quad + \text{Cyclic}\{p_1, p_2, \dots, p_{n-1}, p_n\}, \end{aligned} \quad (4.2)$$

Writing the D -dimensional vector as $\widehat{\ell} = (\ell, \mu, \eta)$, we have

$$\begin{aligned} &\mathcal{A}_L(1^+, 2^+, \dots, k^+, \widehat{\ell}_R^+, -\widehat{\ell}_L^-) \mathcal{A}_R(\widehat{\ell}_L^+, -\widehat{\ell}_R^-, (k+1)^+, \dots, n^+) \\ &= \frac{(\mu^2 + \eta^2)^2}{\langle 1 \ 2 \rangle \cdots \langle k-1 \ k \rangle \langle k+1 \ k+2 \rangle \cdots \langle n-1 \ n \rangle} \\ &\times \frac{[1 | \prod_{i=2}^{k-1} (\mu^2 + \eta^2 - x_{i, \ell_L} x_{\ell_L, i+1}) | k] [k+1 | \prod_{i=k+2}^{n-1} (\mu^2 + \eta^2 - x_{i, \ell_R} x_{\ell_R, i+1}) | n]}{\prod_{i=2}^k (\mu^2 + \eta^2 + x_{\ell_L, i}^2) \prod_{i=k+2}^n (\mu^2 + \eta^2 + x_{\ell_R, i}^2)} \end{aligned} \quad (4.3)$$

The rest two diagrams that contribute are

$$\mathcal{A}_L(\widehat{\ell}_R^+, -\widehat{\ell}_L^-) \mathcal{A}_R(\widehat{\ell}_L^+, -\widehat{\ell}_R^-) \quad \text{and} \quad \mathcal{A}_L(\widehat{\ell}_R^{S_A}, -\widehat{\ell}_L^{S_A}) \mathcal{A}_R(\widehat{\ell}_L^{S_A}, -\widehat{\ell}_R^{S_A})$$

which leads to exactly the same results.

Under the massless conditions of $\widehat{\ell}_L, \widehat{\ell}_R$, we can make the following replacement $\eta^2 \rightarrow \widetilde{\ell}^2$ where $\widetilde{\ell} = (\ell, \mu)$, $\widetilde{\ell} \rightarrow \alpha_L \widetilde{\ell}$ and $\eta \rightarrow \alpha_L \eta$, as well as $\alpha_L = p_L^2/(2\widetilde{\ell} \cdot p_L)$ in succession. After changing η , the general integrand in \mathcal{Q} -cut representation is

$$\begin{aligned} \mathcal{I}^{\mathcal{Q}}(\widetilde{\ell}) &= (2 - 2\epsilon) \sum_{k=2}^{n-2} \frac{1}{\langle 1 \ 2 \rangle \cdots \langle k-1 \ k \rangle \langle k+1 \ k+2 \rangle \cdots \langle n-1 \ n \rangle} \frac{(\mu^2 + \widetilde{\ell}^2)^2 (p_L^2/(2\widetilde{\ell} \cdot p_L))^2}{\widetilde{\ell}^2 (-2\widetilde{\ell} \cdot p_L + p_L^2)} \\ &\times \frac{[1 | \prod_{i=2}^{k-1} (\mu^2 + \widetilde{\ell}^2 - x_{i, \ell_L} x_{\ell_L, i+1}) | k] [k+1 | \prod_{i=k+2}^{n-1} (\mu^2 + \widetilde{\ell}^2 - x_{i, \ell_R} x_{\ell_R, i+1}) | n]}{\prod_{i=2}^k (\mu^2 + \widetilde{\ell}^2 + x_{\ell_L, i}^2) \prod_{i=k+2}^n (\mu^2 + \widetilde{\ell}^2 + x_{\ell_R, i}^2)} \\ &\quad + \text{Cyclic}\{p_1, p_2, \dots, p_{n-1}, p_n\}, \end{aligned} \quad (4.4)$$

where we have summed over helicity states in $(4 - 2\epsilon)$ -dimension (especially including the S_A components in $\dim[\mu] = (-2\epsilon)$ -dimension).

5. Summary

Our results for the series of color-ordered tree amplitudes are very concise. Each of these amplitudes shares a common structure where one part of the amplitude relies on helicity difference of the pair of legs with extra-dimensional momenta and the other part containing pole structures is the same for each case. Our strategies to use the BCFW recursion relations are quite efficient to prove the general expressions for these amplitudes and reveal how different parts of the amplitudes evolve during the recursion.

Conversely, we emphasize that our results are also correct for the associated massive amplitudes, which can be used in the unitarity cut method for 1-loop amplitudes and build superamplitudes on the Coulomb branch. Two of our results $\{\ell_2^-, \ell_1^-\}$ and $\{\ell_2^-, \ell_1^{S_a}\}$ are completely new.

Using these tree amplitudes we calculate the general 1-loop all-plus integrand with any numbers of gluons by using \mathcal{Q} -cut construction. This integrand also has a good structure.

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